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Construction of quantum states from an optimally truncated von Neumann lattice of coherent states

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Abstract. Arbitrary quantum states can be expanded with a very good accuracy in terms of coherent states on a truncated von Neumann lattice. Optimization techniques are used to further reduce the size of the basis, whilst still preserving the same degree of accuracy. Various p -norms are adopted as accuracy measures, which reflect different practical needs. Numerical examples demonstrate that a significant reduction in the size of the basis can be achieved with optimization techniques, without substantial loss of accuracy.

1. Introduction

One of the most important properties of coherent states is that they can be used as an overcomplete basis in the Hilbert space. We consider the harmonic oscillator Hilbert space H spanned by the number eigenstates $\{|N\rangle; N = 0, 1, 2, \dots\}$ and the coherent states

$$|A\rangle = \exp\left(-\frac{|A|^2}{2}\right) \sum_{N=0}^{\infty} \frac{A^N}{(N!)^{1/2}} |N\rangle = D(A)|0\rangle \quad (1)$$

$$D(A) = \exp[Aa^\dagger - A^*a] \quad (2)$$

where a^\dagger, a are the usual creation and annihilation operators. $D(A)$ is the displacement operator associated with the Heisenberg–Weyl group whose generators form the algebra $[a, a^\dagger] = \mathbf{1}$. The resolution of the identity for these coherent states is

$$\int \frac{d^2A}{\pi} |A\rangle\langle A| = I \quad (3)$$

and can be used to expand an arbitrary (normalized) state $|f\rangle$ as

$$|f\rangle = \int \frac{d^2A}{\pi} f(A) |A\rangle \quad f(A) = \langle A|f\rangle. \quad (4)$$

The above expansion uses all coherent states in the complex plane. It is well known that this set of coherent states is highly overcomplete and that there are much smaller subsets of coherent states which are also overcomplete. A well known overcomplete set of coherent states is the von Neumann lattice. This is the set of coherent states

$$|A_{MK}\rangle = |\alpha M + i\beta K\rangle \quad (5)$$

where M, K are integers and $S = \alpha\beta$ is the lattice area. It is well known [1] that this set is overcomplete if $S < \pi$; and undercomplete if $S > \pi$.

In a recent paper [2], a truncated von Neumann lattice of coherent states was used for an approximate construction of various quantum states. For any quantum state $|f\rangle$, many of the coherent states on a (full) von Neumann lattice are very far in phase space; and their overlap with $|f\rangle$ is very small. Neglecting them, we get a truncated von Neumann lattice, which from a practical point of view is much easier to handle, and which is sufficient for an accurate reconstruction of $|f\rangle$. It has been found that this expansion is both accurate and robust (in the sense that random noise in the coefficients weakly affects the constructed state).

More specifically, a truncated von Neumann lattice of coherent states is the set of states of equation (5) with (M, K) a pair of integers which take values in a set I (which is a finite subset of $Z \times Z$). We call I' the set of the rest of the values of the pair of integers (M, K) (i.e. $I' = Z \times Z - I$). In order to have a good approximation, the set I should be chosen in such a way that for all $(M, K) \in I'$ we have:

$$\langle f|A_{MK}\rangle \ll \mathbf{1}. \quad (6)$$

One way of achieving this is by calculating the quantities $\langle x \rangle, \langle p \rangle, \Delta x, \Delta p$, where as usual:

$$\langle x \rangle = \langle f|\hat{x}|f\rangle \quad (7)$$

$$\langle x^2 \rangle = \langle f|x^2|f\rangle \quad (8)$$

$$\Delta x = [\langle x^2 \rangle - \langle x \rangle^2]^{1/2} \quad (9)$$

and similar definitions hold for the momentum p . The set I contains all the values of M, K such that

$$\langle x \rangle - \mu(\Delta x) < \alpha M < \langle x \rangle + \mu(\Delta x) \quad (10)$$

$$\langle p \rangle - \mu'(\Delta p) < \beta K < \langle p \rangle + \mu'(\Delta p) \quad (11)$$

where μ, μ' are positive numbers. Clearly, the bigger the μ, μ' are, the better the approximation.

On the other hand, an expansion is practically useful if it can reconstruct the original state accurately, with *a few* terms. While performing the numerical work presented in [2], it was found that ‘cleverly selected’ subsets of I (i.e. fewer coherent states) give results which are almost as good as those with the full set I . In this paper we present some interdisciplinary research, using optimization techniques to find these ‘cleverly selected’ subsets of I . The optimization reconciles the contradictive requirements of having only a few terms and at the same time gives good accuracy. There are many powerful computer programs available which perform general purpose optimization, but clearly they have not been written for our own problem. In order to use them, we need to formulate our problem in their language.

In section 2 we rigorously express our problems in the language of optimization theory. Mathematical formulations of these problems are derived using various p -norms for various definitions of accuracy, appropriate to different practical needs. Having done that, in section 3 we are able to use existing optimization programs in our own context. Numerical results are presented in section 4. In section 5 we conclude with a discussion of our results and suggestions for applications to other areas.

2. The optimization problem

In this section we express our problem in the language of optimization theory and explicitly define the optimization problem that we solve.

We start with a general optimization problem which can be formulated as follows. Given a complex function $f(x)$ where $x \in R$, and a discrete set of N ‘basis’ functions $\mathbf{A}(x)$ where $\mathbf{A}(x) = (A_1(x), A_2(x), \dots, A_N(x))^T$ (which are not necessarily orthogonal) find appropriate weights $\mathbf{f} = (f_1, f_2, \dots, f_N)^T$ which minimize the error of approximation:

$$\Omega(\mathbf{f}) = \left\| f(x) - \sum_{i=1}^N f_i A_i(x) \right\|_p \quad (12)$$

such that the following bounds are satisfied:

$$x^L \leq x \leq x^U \quad \text{range of the free variable} \quad (13)$$

$$\mathbf{f}^L \leq \mathbf{f} \leq \mathbf{f}^U. \quad \text{bounds of the basis weight variables.} \quad (14)$$

The above formalism is general, but for our purposes N is expressed in terms of the lattice size. The norm in equation (12) is defined as

$$\|g(x)\|_p = \left[\int_{x^L}^{x^U} dx |g(x)|^p \right]^{1/p}. \quad (15)$$

As p increases, the ‘worst points’ (i.e. the points x at which the difference $|f(x) - \sum_{i=1}^N f_i A_i(x)|$ is large) contribute more in the evaluation of $\Omega(\mathbf{f})$. In the case $p \rightarrow \infty$,

$$\|g(x)\|_\infty = \max_x |g(x)| \quad (16)$$

and

$$\Omega(\mathbf{f}) = \max_x \left| f(x) - \sum_{i=1}^N f_i A_i(x) \right|. \quad (17)$$

In this case the above problem is a min–max optimization problem.

The value of p depends on the application. Small values of p are appropriate for applications where the constructed quantum state should be ‘on average’ close to the desired quantum state, and where large deviations in small regions of x are of no particular importance. Large values of p are appropriate for applications where large deviations of the constructed quantum state from the desired one, even at small regions of x , can be catastrophic; while small deviations are relatively harmless. In a more general context, error is the difference of the approximation from the exact result, to a power p . Large values of p put a lot of weight to large differences and are suitable for applications where large deviations from the exact result are very harmful and very undesirable.

2.1. Use of a smaller basis

In several applications, the available ‘basis’ set is more than enough for the required accuracy, and it is desirable, for reasons of simplicity and cost efficiency, to use just a small subset of the original set. In this case, the objective is not only to choose optimally the weights \mathbf{f} , but to enforce some of them only to be present, setting all others to zero.

It is noted that this choice should have no consequence on the formulation of the bounds defined in equation (14), rather it will simply indicate that those bounds for non-participating basis functions are set to zero, while for participating ones they are left free as defined.

To derive the general formulation for this problem, the number of participating basis functions is defined to be $\bar{N} \leq N$ and the optimal solution should include at most \bar{N} basis functions in order to minimize any desired norm of the fitting error. In order to do this we introduce the variables y_i ($i = 1, 2, \dots, N$) which take the values one or zero according to whether the basis function $A_i(x)$ will be included or not included in the

expansion, correspondingly. To accommodate the corresponding bounds on the weights f_i a transformation $w_i = y_i f_i$ is included in the formulation, where w_i will replace the corresponding weights f_i in the norm. The new optimization (variational) problem is to minimize the function

$$\min_{y,w} \Omega(\mathbf{w}) = \min_{y,w} \left\| f(x) - \sum_{i=1}^N w_i A_i(x) \right\|_p \quad (18)$$

subject to the bounds on x

$$x^L \leq x \leq x^U \quad (19)$$

and the bounds on the weights w_i

$$y_i f_i^L \leq w_i \leq y_i f_i^U \quad i = 1, 2, \dots, N. \quad (20)$$

It is clear from what has been said above that

$$\sum_{i=1}^N y_i = \bar{N}. \quad (21)$$

The formulation presented in the model of equations (18)–(21) is the most general formulation for the optimal choice of a subset of basis functions from a larger finite set. It makes no assumptions regarding the norm used to express the objective function $\Omega(\cdot)$ nor regarding the nature of the variable x . Although a single free variable was chosen for simplicity, it is always possible to include a vector of free variables (relevant to the case of multimode quantum states). The reformulation of approximation problems as mathematical programming models is known (e.g. [3]). [4] presented the use of a mixed-integer nonlinear formulation to optimally select the parameters, and structure, of a model fitting experimental data with applications to infrared spectroscopy. This work extends these ideas by generalizing the error norms used, and introduces the application of this theory to quantum state engineering.

2.2. Discretization of the variable x

In many practical cases the function $f(x)$ is obtained experimentally at discrete values of the variable x . Even if an analytic form of the $f(x)$ is available, discretization of the variable x might be necessary for numerical purposes (if the above minimization cannot be performed analytically, being a variational problem in the general context). In these situations some modification of the previous model is needed in order to take into account the discrete nature of x . The set of values of x is considered to contain N_x points, such that now $\mathbf{x} = (x_1, x_2, \dots, x_{N_x})^T$. These points are implicitly assumed to satisfy the bounds on variable x . The new optimization problem is again described with equations (18)–(21) but the p -norm which above was defined in terms of an integral, is here defined in terms of the sum

$$\|g(x)\|_p = \left[\sum_j |g(x_j)|^p (x_j - x_{j-1}) \right]^{1/p}. \quad (22)$$

In the case of a constant ‘discretization step’ h , the above equation becomes

$$\|g(x)\|_p = \left[h \sum_j |g(x_j)|^p \right]^{1/p}. \quad (23)$$

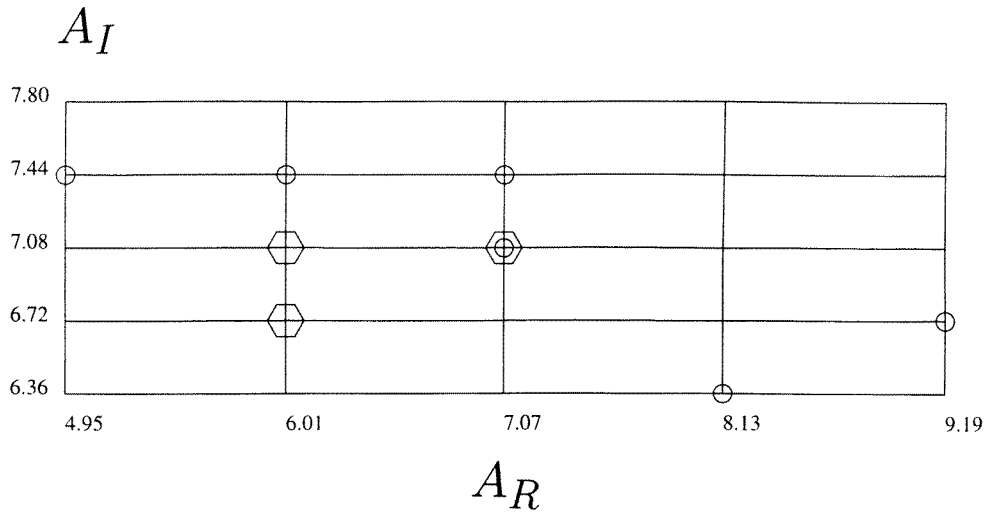


Figure 1. The 5×5 truncated von Neumann lattice used for the construction of the squeezed state described in the text. The optimally selected states are shown with a hexagon for the case of three coherent states and with a circle for the case six coherent states.

3. Computational solution

The models formulated for the three norm cases $p = 1$, $p = 2$, and $p = \infty$ represent mixed integer mathematical programming formulations. A standard approach applicable to such problems is the use of the well known ‘branch and bound’ algorithms. These are efficient algorithms that allow the search of the combinatorial tree of binary variables (or generally integer variables) in such a way that the computational effort is usually far better than the worst case (combinatorial complexity). An excellent survey of methods and formulation applicable to mixed integer-linear programming problems can be found in [5]. The case of mixed integer-nonlinear programming required to address the case of $p = 2$ is addressed through special algorithms. A review of these can be found in [6].

All runs were carried out using the GAMS mathematical programming package-language [7], interfacing to the solvers CPLEX (mixed integer-linear programming solver, using the simplex algorithm with branch and bound on the binary variables, DICOPT++ (mixed integer-nonlinear programming solver using the CPLEX solver and MINOS solver iteratively), and MINOS (nonlinear programming solver).

4. Examples

As an example we have considered the squeezed state

$$|A; r, \theta\rangle = S(r, \theta)|A\rangle \tag{24}$$

$$S(r, \theta) = \exp[\frac{1}{4}re^{-i\theta}(a^\dagger)^2 - \frac{1}{4}re^{i\theta}a^2] \tag{25}$$

with $A = 5\sqrt{2}(1+i)$, $r = 0.916$, $\theta = 0$. This state has been reconstructed using subsets of the 5×5 truncated von Neumann lattice shown in figure 1. Optimal selection of three coherent states and six coherent states from the 25 coherent states in this lattice produced the results shown in figures 2 and 3 correspondingly, using the norm $p = 1$ (dashed curve), $p = 2$ (star curve), and $p = \infty$ (dash-dot curve). The original state is also shown for comparison (solid

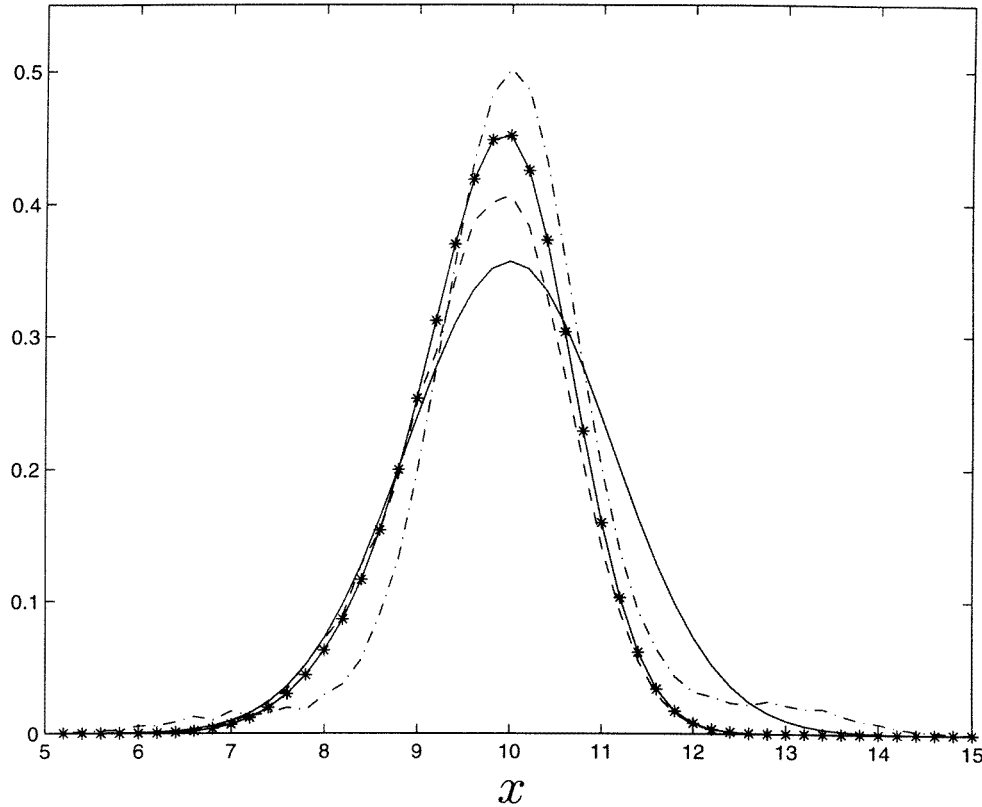
$$P(x)$$


Figure 2. Construction of the squeezed state described in the text (solid curve) with three optimally selected coherent states from the truncated von Neumann lattice shown in figure 1. Three norms have been used: $p = 1$ (dashed curve); $p = 2$ (star curve); and $p = \infty$ (dash-dot curve).

curve). The optimally selected states are shown in figure 1 with a hexagon for case of three coherent states and a circle for the case of six coherent states.

Figure 4 shows the error of the approximation as a function of the number of coherent states optimally selected from the 5×5 lattice shown in figure 1. The error of the approximation is defined as

$$e = h \sum_{j=1}^{N_x} e_j \quad (26)$$

where Δx is the discretisation interval i.e. the range of calculation divided by the number of points (in our case $h = 0.2$, $N_x = 50$). It is seen that as the number of optimally selected coherent states is reduced from 25 to 12 or 9, the loss of accuracy is insignificant (the error goes from 0.15 to 0.18 and 0.19, correspondingly). As the number of optimally selected coherent states is reduced even more, the error increases, quickly. This graph demonstrates the need for an optimal selection of the basis since in this example the approximation with six coherent states is almost as good as with 25 coherent states.

Our method uses an $M \times K$ truncated von Neumann lattice of $N = MK$ coherent states

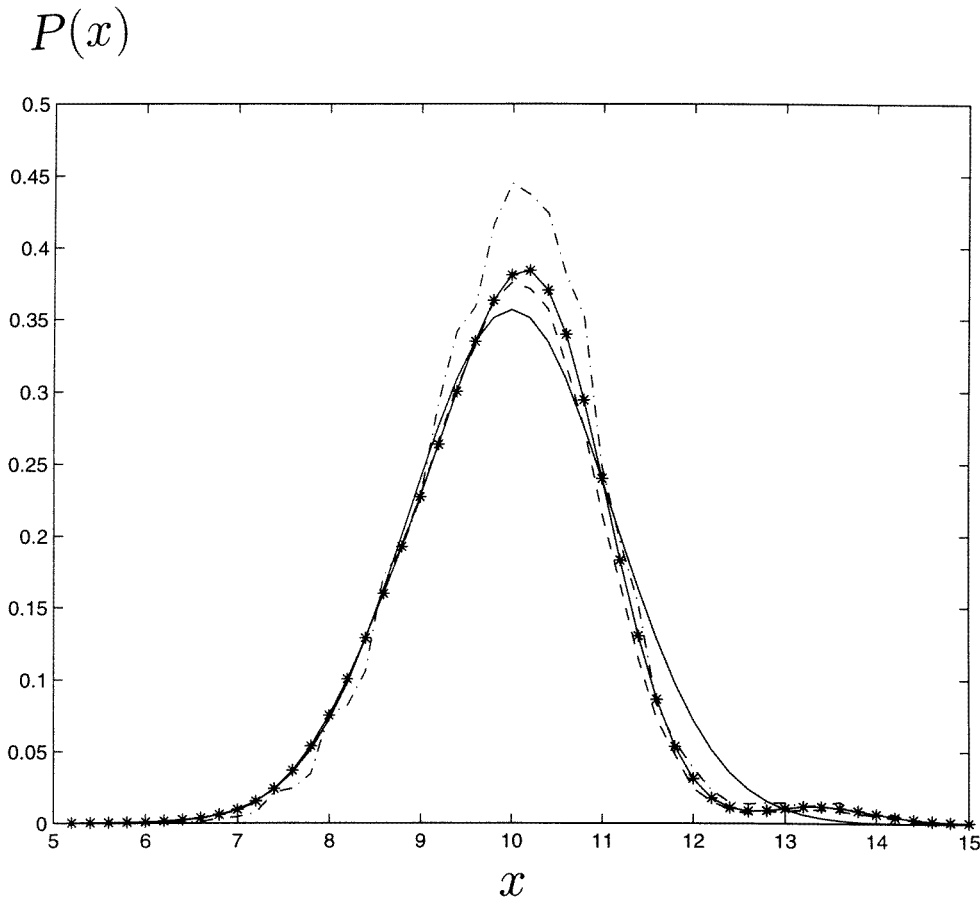


Figure 3. Construction of the squeezed state described in the text (solid curve) with six optimally selected coherent states from the truncated von Neumann lattice shown in figure 1. Three norms have been used: $p = 1$ (dashed curve); $p = 2$ (star curve); and $p = \infty$ (dash-dot curve).

and optimally selects \bar{N} of them. In the above examples we had fixed N ($M = K = 5$) and varied \bar{N} . In figure 5 we compare two cases with varying N ($N = 25$ and $N = 9$) for fixed \bar{N} ($\bar{N} = 3$). The 5×5 von Neumann lattice is the one shown in figure 1, while the 3×3 truncated von Neumann lattice is

$$A_{mn} = 2.12m + 0.71ni + (4.95 + 6.36i) \quad m = 0, 1, 2 \quad n = 0, 1, 2. \quad (27)$$

The case $N = 9$ is shown by the dashed curve; the case $N = 25$ is shown by the dash-dotted curve; and the original state is shown in the solid curve. The norm $p = 1$ has been used.

It is seen that although the *same* number of coherent states has been used in both cases, the fact that they have been optimally selected from different lattices led to different results. For a given number of coherent states, optimal selection from a large lattice produces better results.

In order to clearly show the effect of the optimization procedure, we point out that in the case $N = 9$ and $\bar{N} = 3$ discussed above, the program selects the states $(m = 0, n = 0)$, $(m = 0, n = 1)$ and $(m = 1, n = 1)$ from the lattice of equation (27), and attaches the coefficients $f_{00} = -0.1987 - 0.2124i$, $f_{01} = -0.2800 - 0.2925i$ and

error

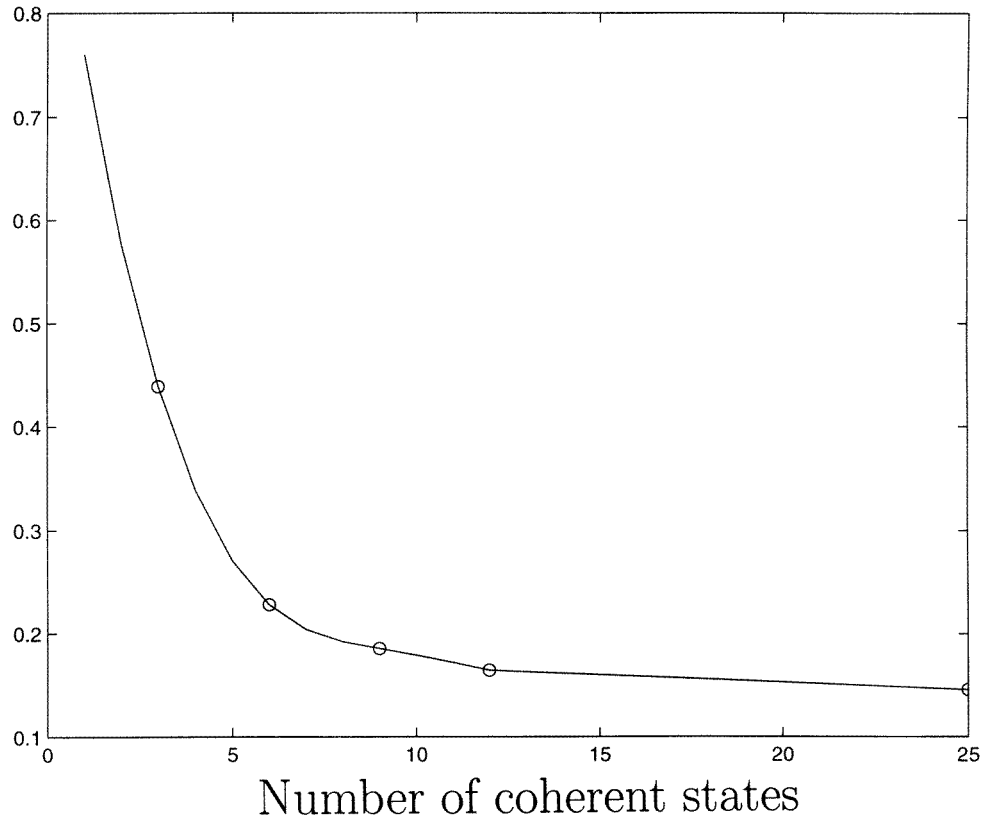


Figure 4. Error (equation (26)) as a function of the number of coherent states optimally selected from the 5×5 lattice shown in figure 1. The $p = 1$ norm has been used.

$f_{11} = 0.8860 + 0.8843i$ correspondingly. For comparison we note that equation (4) gives the values $f_{00} = -0.0676 + 0.1837i$, $f_{01} = -0.2123 - 0.1817i$ and $f_{11} = 1.0112$ correspondingly, for these coefficients.

Some of the norm 2 results ($p = 2$) have been difficult to converge with the solvers used. This can be attributed to the number of conditions in the least-squares problems and the use of a standard nonlinear optimization solver (MINOS) to solve these subproblems within the mixed integer-nonlinear programming solver algorithm. The other norm formulations ($p = 1, \infty$) which we investigated are much more stable and easier to solve. This observations also explain why the norm 2 results appear to be suboptimal, although from the theoretical point of view the formulation is convex and hence guaranteed to have a unique globally optimal solution.

5. Discussion and applications to other areas

One of the most important properties of coherent states is the resolution of the identity of equation (3). Using it, we can expand an arbitrary state in terms of coherent states as in equation (4). This expansion is exact but contains all coherent states in the complex plane. Since this basis is highly overcomplete, it is desirable to reduce its size. In [8] the coherent

$$P(x)$$

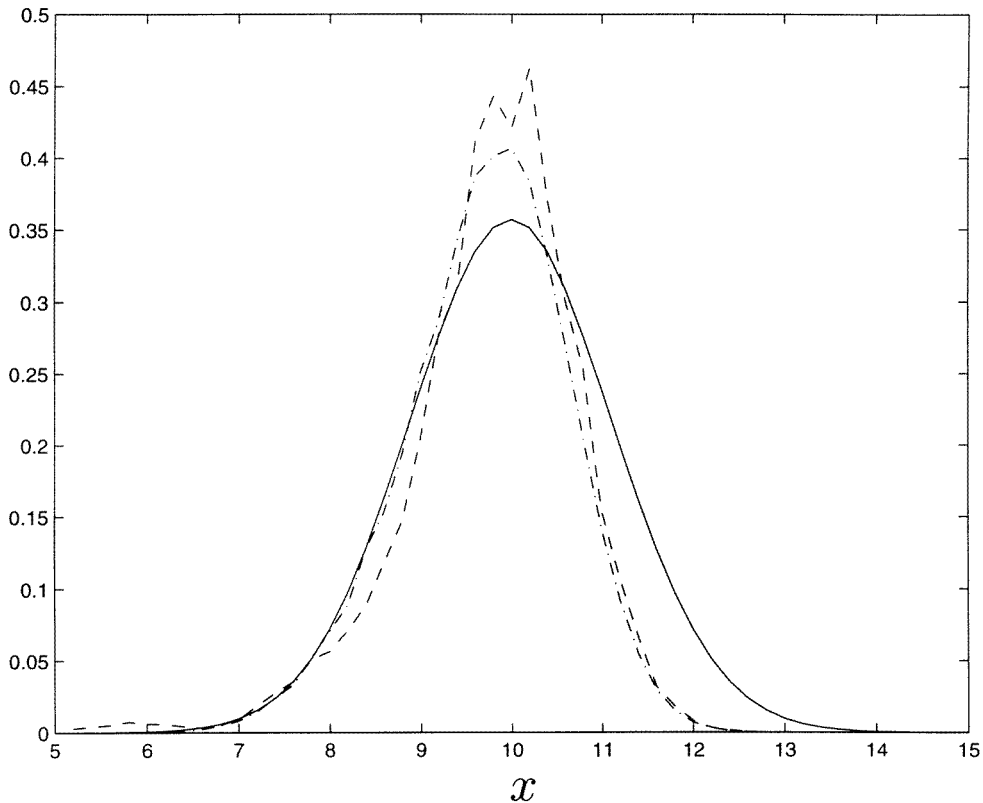


Figure 5. Construction of the squeezed state described in the text (solid curve) using three coherent states optimally selected from: the 5×5 lattice of figure 1 (dash-dot curve); and the 3×3 lattice of equation (27) (dashed curve). The $p = 1$ norm has been used.

states on a line in the complex plane have been considered and resolutions of the identity in terms of these states have been studied. Using them, we can express an arbitrary state as a line integral of coherent states. This is one approach within the more general framework of quantum state engineering [9], the aim of which is to construct arbitrary quantum states.

Superpositions of coherent states have been constructed experimentally (in the context of Schrödinger cats). However, if the method of expanding arbitrary states in terms of coherent states, is going to be useful for quantum state engineering, it needs to employ only a few coherent states. The expansion of [8] uses an infinite number of coherent states.

In [2], a truncated von Neumann lattice has been used for an accurate construction of various quantum states. It is practically very important to use small bases in these expansions and at the same time get accurate results. Optimization techniques reconcile these contradictory requirements. In this paper, we have translated these problems in a language suitable for the use of existing optimization techniques and numerical programs. We have found that through optimization methods a considerable reduction of the size of the basis can be achieved with effectively no loss in accuracy. Figure 4 demonstrates this point clearly. As the size of the basis decreases, the error increases very little; up to a certain point (in this case six coherent

states) below which the error increases significantly.

We have presented our ideas in the context of quantum states in quantum mechanics. However, their applicability is much more general to any problem where an expansion of an arbitrary function in terms of Gaussian functions (our coherent states) or other localized functions (e.g. wavelets), is useful. An example is the area of signal processing where the expansion of an arbitrary signal in terms of Gaussian signals is known as Gabor expansion [10] and is used widely.

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